

The Length of an SLE—Monte Carlo Studies

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Abstract The scaling limits of a variety of critical two-dimensional lattice models are equal to the Schramm–Loewner evolution (SLE) for a suitable value of the parameter κ . These lattice models have a natural parametrization of their random curves given by the length of the curve. This parametrization (with suitable scaling) should provide a natural parametrization for the curves in the scaling limit. We conjecture that this parametrization is also given by a type of fractal variation along the curve, and present Monte Carlo simulations to support this conjecture. Then we show by simulations that if this fractal variation is used to parametrize the SLE, then the parametrized curves have the same distribution as the curves in the scaling limit of the lattice models with their natural parametrization.

Keywords Schramm–Loewner evolution · Natural parametrization · Fractal variation

1 Introduction

Schramm–Loewner evolution (SLE) is a one parameter family of random processes that produce random curves in the plane [19]. (When we refer to the curves of SLE we will mean the trace of the SLE.) Beffara [2, 3] proved that the Hausdorff dimension of the SLE curve is $1 + \kappa/8$ a.s. for $\kappa \leq 8$. With the usual definition of length, the length of these SLE curves is infinite. Nonetheless, one would like to define some notion of the length of an SLE curve.

To motivate our proposal for a definition of length for SLE we first consider models which come from scaling limits of models on a lattice, e.g., the loop-erased random walk (LERW), the self-avoiding walk (SAW), interfaces in the critical Ising model, and the percolation exploration process. Before the scaling limit, the random curves in the lattice model (which are walks or interfaces) have a natural parametrization arising from the number of steps in the curve, or equivalently the length of the curve. This leads to a parametrization of the curves in the scaling limit that we will refer to as the “natural parametrization.” It is defined as follows. Let $W(n)$ be a random curve in the lattice model with the lattice spacing

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equal to 1. The mean square distance the curve travels after n steps should grow as n^{2p} for some critical exponent p

$$E[|W(n) - W(0)|^2] \approx cn^{2p}. \tag{1}$$

The exponent p should be related to the Hausdorff dimension, d_H , of the curve by $p = 1/d_H$. From now on we will assume this is the case and use $1/d_H$ in place of p .

Initially, $W(n)$ is defined only for non-negative integers n . We extend $W(t)$ to all positive real t by linearly interpolating. Then we define

$$\omega(t) = \lim_{n \rightarrow \infty} n^{-1/d_H} W(nt). \tag{2}$$

We will refer to the resulting parametrization of the curves in the scaling limit as the natural parametrization. This definition is analogous to the construction of Brownian motion as the scaling limit of the ordinary random walk. The goal of this paper is to propose a parametrization of SLE which corresponds to the natural parametrization of the scaling limits of the various lattice models and to provide some support from simulations for this correspondence.

To introduce our parametrization of SLE, let $0 = t_0 < t_1 < t_2 \cdots < t_n = t$ be a partition of the time interval $[0, t]$. The usual definition of the length or total variation of the random curve γ over the time interval $[0, t]$ would be the supremum over all such partitions of

$$\sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})|. \tag{3}$$

The length of $\gamma(t_j) - \gamma(t_{j-1})$ will be of order $|t_j - t_{j-1}|^{1/d_H}$, and since $d_H > 1$, the total variation of γ will be infinite. Since the length of a segment is of order $|t_j - t_{j-1}|^{1/d_H}$, this suggests that we consider the quantity,

$$fvar(\gamma[0, t], \Pi) = \sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})|^{d_H} \tag{4}$$

where Π denotes the partition, $\{t_0, t_1, \dots, t_n\}$. Then we take a sequence of partitions Π_m whose mesh converges to zero. (The mesh of a partition is the length of the largest subinterval.) Then we consider the limit,

$$\lim_{m \rightarrow \infty} fvar(\gamma[0, t], \Pi_m). \tag{5}$$

Of course, the existence of this limit and its dependence on the sequence of partitions used is a subtle question.

This quantity is sometimes called the p -variation in the stochastic processes literature. (p is used where we have d_H .) We find the name p -variation particularly dull, and will refer to this quantity as the fractal variation. When $d_H = 2$, this is the quadratic variation studied by Lévy for Brownian motion [16]. It is non-random and proportional to t under suitable conditions on the convergence of the sequence of partitions [5, 6].

A serious drawback of this definition is that it depends on the parametrization of the curve. For example, in the scaling limit of a lattice model we could compute this fractal variation using either the parametrization of the curve by capacity or its natural parametrization.

In our setting the dependence of the above definition on the choice of parametrization is particularly troubling since our goal in introducing this quantity is to define a parametriza-

tion. A better definition that does not depend on the choice of parametrization of the curve is the following. Let $\Delta t > 0$. We define t_i inductively. Let $t_0 = 0$. Given t_{i-1} , let t_i be the first time after t_{i-1} with

$$|\gamma(t_i) - \gamma(t_{i-1})| = (\Delta t)^{1/d_H}.$$

Let n be the last index with $t_n < t$. Then we will define

$$fvar(\gamma[0, t], \Delta t) = n \Delta t = \sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})|^{d_H} \tag{6}$$

and define the fractal variation of γ over $[0, t]$ to be

$$fvar(\gamma[0, t]) = \lim_{\Delta t \rightarrow 0} fvar(\gamma[0, t], \Delta t). \tag{7}$$

In Fig. 1 an SLE curve is shown for $\kappa = 8/3$. There are two copies of the same curve. In the copy on the left it is divided into segments which correspond to equal changes in the capacity, while in the copy on the right it is divided into segments of equal fractal variation. When capacity is used to determine the segments, the segments appear to have varying lengths, while the segments determined using the fractal variation appear to have equal length.

In the next section we will consider this fractal variation in several lattice models, and present Monte Carlo simulations that show the limit in (7) converges to a non-random constant. In Sect. 3 we use Monte Carlo simulations to compare random variables defined using the natural parametrization in the lattice models with the corresponding random variables for SLE curves parametrized by their fractal variation. Section 4 gives some details about our simulations of SLE and the various lattice models.

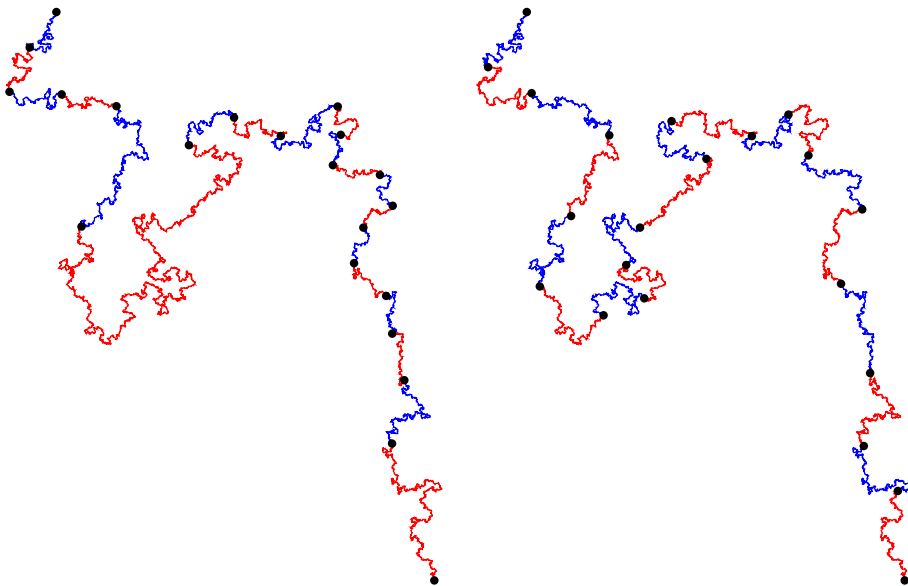


Fig. 1 (Color online) An $SLE_{8/3}$ curve divided into equal segments according to parametrization by capacity (left) and by fractal variation (right)

2 Fractal Variation of Discrete Models

We begin with our main conjecture:

Conjecture 1 Let $\gamma(t)$ be a random curve in the scaling limit of a critical lattice model with parametrization given by the length of the lattice curve suitably scaled (the natural parametrization). Then the fractal variation of $\gamma[0, t]$ exists and is proportional to t . (The constant of proportionality will depend on the lattice.)

The fractal variation of $\gamma[0, t]$ is a priori a random variable. Part of the conjecture is that this random variable is not random. Note that for models which are defined in the half plane, the scaling limit is expected to be invariant under dilations. This implies that for models in the half plane, if the fractal variation is nonrandom, then it must be proportional to the natural parameterization. In this section we will support the conjecture that the fractal variation is not random by numerically computing the fractal variation for the LERW, the SAW, Ising interfaces and percolation interfaces at a fixed value of their natural parameterization. The simulations of the next section will test the conjecture that this fractal variation is proportional to the natural parameterization. More details on the definition of these models and our simulations may be found in Sect. 4.

The fractal variation is the limit as $\Delta t \rightarrow 0$ of $fvar(\gamma[0, t], \Delta t)$. The quantity $fvar(\gamma[0, t], \Delta t)$ is a random variable. For a particular lattice model we simulate this random variable for several values of Δt , and for each value plot the cumulative distribution function, i.e., $P(fvar(\gamma[0, t]) \leq x)$ as a function of x . If the fractal variation is indeed constant, then these cumulative distribution functions should converge to a function that is 0 left of the value $fvar(\gamma[0, t])$ and 1 to the right.

Figure 2 shows these cumulative distribution functions for the LERW for several values of Δt . Note that the range of the horizontal axis shown is rather narrow and does not include zero. We should caution that we cannot take Δt too small. It must be large enough that the number of steps in the lattice walk corresponding to a single Δt is large. If Δt is extremely small, there will be only a single step in the lattice walk corresponding to each Δt and $fvar(\gamma[0, t], \Delta t)$ will just equal $n\Delta t$ where n is the number of steps in the lattice walk. Thus as $\Delta t \rightarrow 0$, we will first see $fvar(\gamma[0, t], \Delta t)$ converging to a constant, but then there will be a crossover where it begins to converge to a different constant. The beginning of this crossover is seen in Fig. 2 where the curve with the smallest value of Δt , 2×10^{-4} , is shifted to the right of the other curves.

For the SAW the cumulative distributions of $fvar(\gamma[0, t], \Delta t)$ are shown in Fig. 3. For the SAW it is possible to simulate walks with a very large number of steps (one million). This is the reason one does not see any shift of these curves even for the smallest Δt of 10^{-4} .

For interfaces in the critical Ising model, Fig. 4 shows the cumulative distributions of $fvar(\gamma[0, t], \Delta t)$. The smallest value of Δt shown is 5×10^{-4} , larger than the smallest value plotted for the LERW or SAW. One can already see the curve shifting to the left for this relatively large value of Δt . The Ising model is the most difficult to simulate since it is a truly two dimensional model. So the interfaces we can generate are not nearly as long as for the other lattice models.

The percolation interface is the easiest of the four lattice models to simulate, and we can generate curves with four million steps. However, this model has the largest Hausdorff dimension of the lattice models, and this makes the finite lattice effects occur at relatively large values of Δt . Figure 5 shows the cumulative distributions of $fvar(\gamma[0, t], \Delta t)$. For

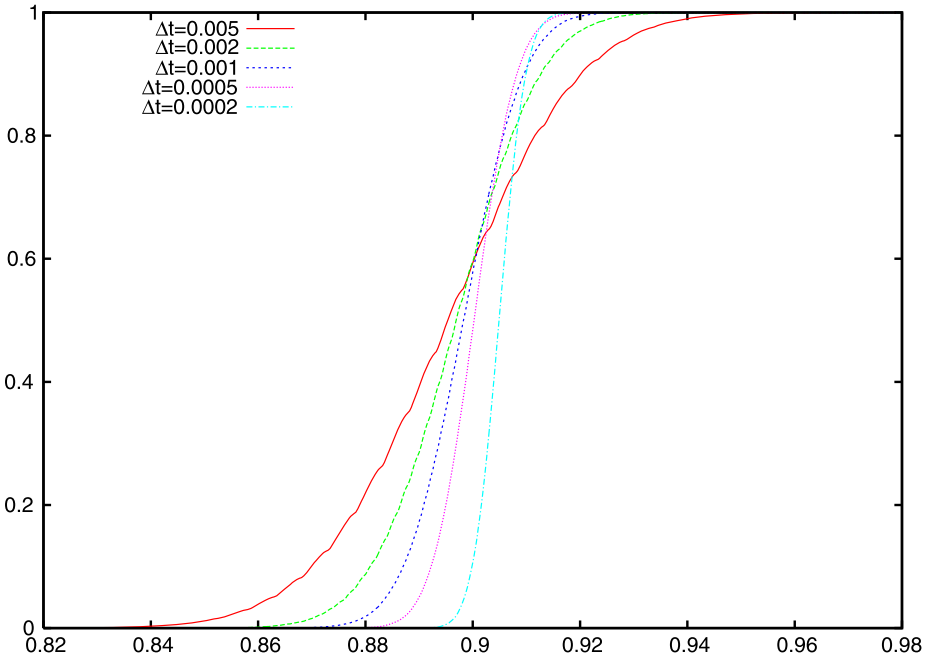


Fig. 2 (Color online) The cumulative distribution of the random variable $fvar(\gamma[0, t], \Delta t)$ for the LERW for a fixed t and several values of Δt

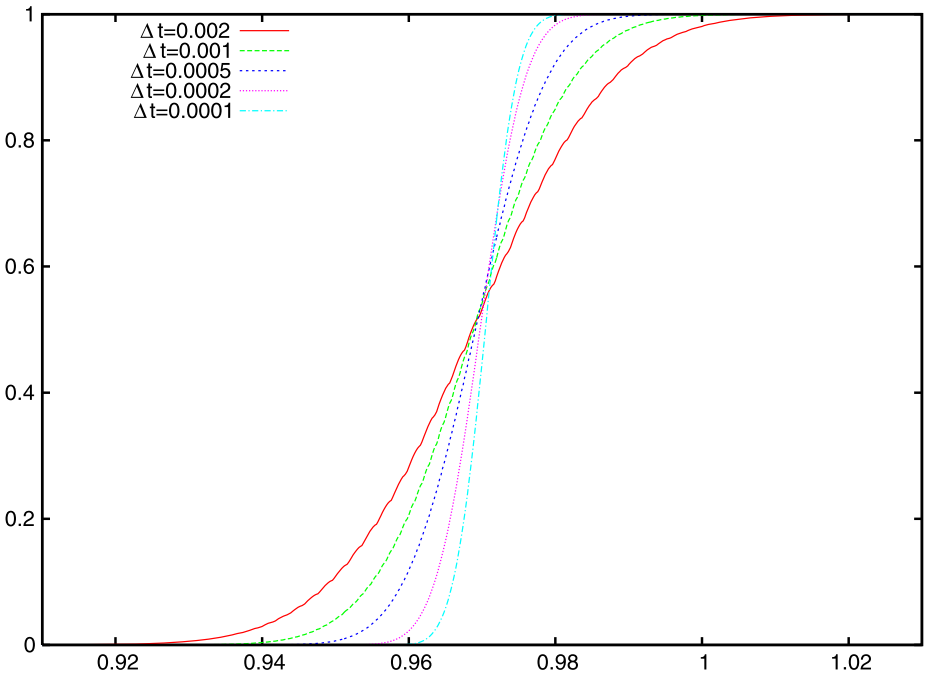


Fig. 3 (Color online) The cumulative distribution of the random variable $fvar(\gamma[0, t], \Delta t)$ for the SAW for a fixed t and several values of Δt

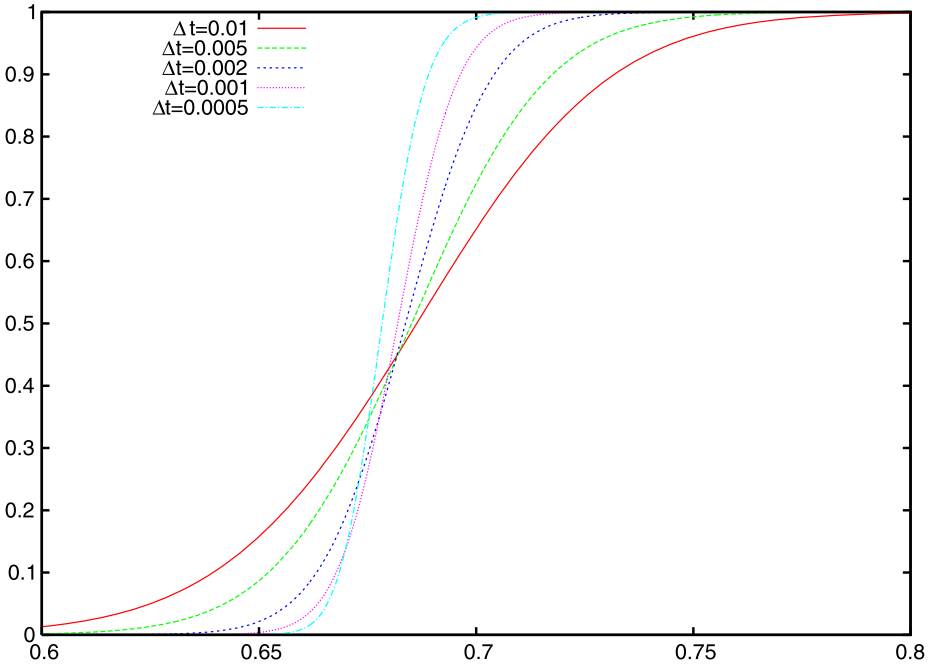


Fig. 4 (Color online) The cumulative distribution of the random variable $fvar[\gamma[0, t], \Delta t]$ for the Ising model for a fixed t and several values of Δt

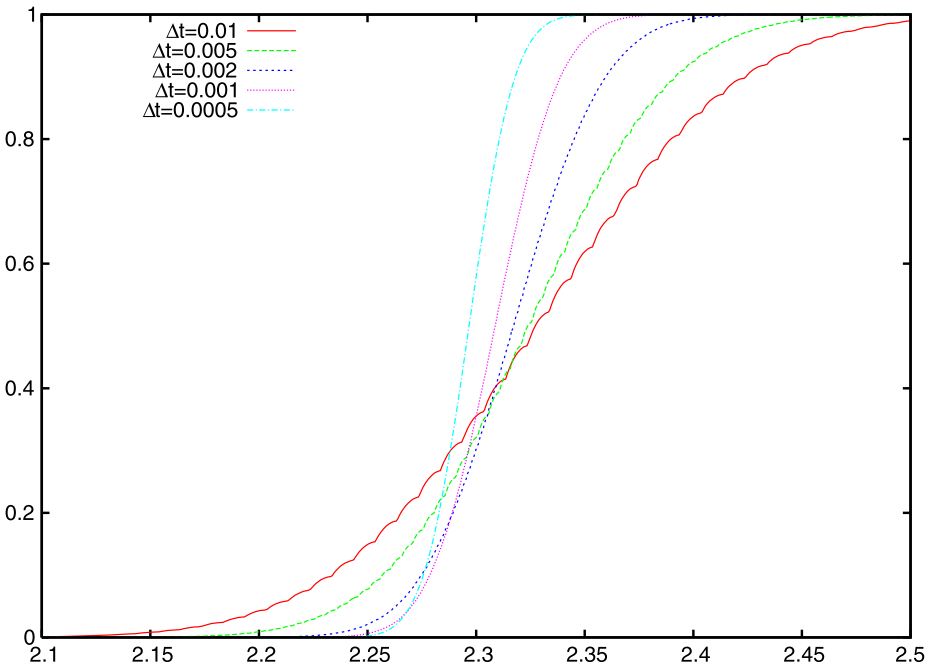


Fig. 5 (Color online) The cumulative distribution of the random variable $fvar[\gamma[0, t], \Delta t]$ for percolation for a fixed t and several values of Δt

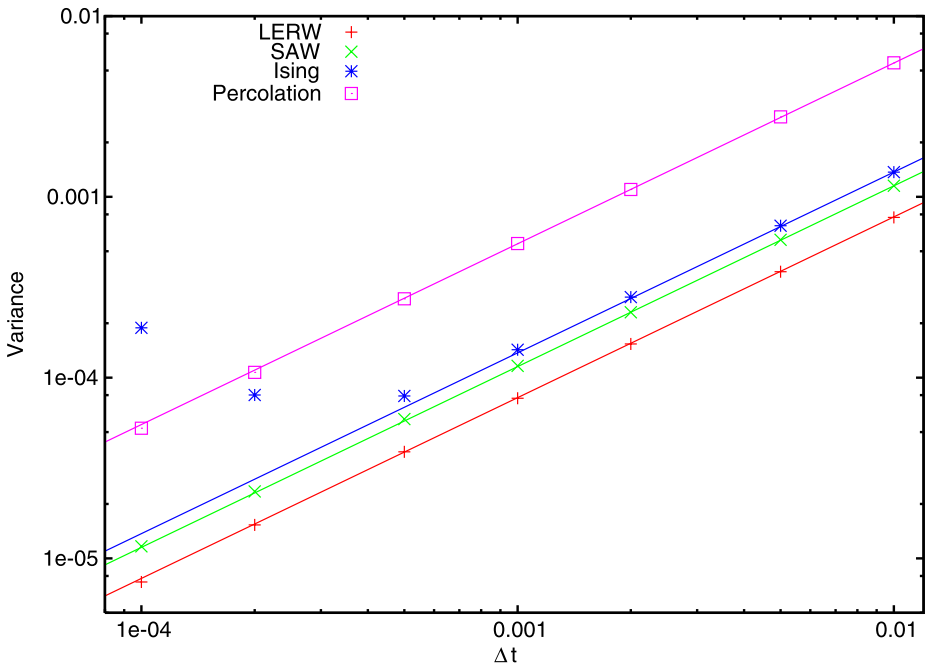


Fig. 6 (Color online) For each of the four lattice models we plot the variance of the random variable $fvar(\gamma[0, t], \Delta t)$ as a function of Δt

$\Delta t = 5 \times 10^{-4}$, the smallest value shown, one can see the curve shifting to the left. In fact, this shift is just barely visible for $\Delta t = 10^{-3}$.

Figures 2, 3, 4 and 5 indicate that the random variables $fvar(\gamma[0, t], \Delta t)$ are converging to a constant as Δt goes to zero. To study this quantitatively, we plot the variance of these random variables as a function of Δt in Fig. 6. We plot these variances for all four lattice models. In all four cases we also draw a line with slope 1 which attempts to fit the data for the values of Δt before the crossover behavior sets in. This linear fit is extremely good until Δt becomes small enough that the finite lattice effects are significant. Note that more values of Δt are shown in Fig. 6 than were shown in the plots of the cumulative distributions of $fvar(\gamma[0, t], \Delta t)$. The deviation of the data points in Fig. 6 from the corresponding straight line begins roughly at the value of Δt where one can see the plots of the cumulative distribution of $fvar(\gamma[0, t], \Delta t)$ beginning to shift in Figs. 2, 3, 4 and 5.

Since the data in Fig. 6 is well fit by a line with slope 1, this indicates the variance of $fvar(\gamma[0, t], \Delta t)$ goes to zero as Δt . Note that the number of terms in the sum defining $fvar(\gamma[0, t], \Delta t)$ is of the order of $1/\Delta t$. For the average of N i.i.d. random variables, the variance also goes to zero as $1/N$. This suggests that the convergence of $fvar(\gamma[0, t], \Delta t)$ to a constant is some form of a law of large numbers. The random variables being summed are not independent, but one can hope that their correlations decay in some suitable way.

3 Comparison of SLE and the Lattice Models

The theorems and conjectures that state that the scaling limit of some discrete model is SLE_κ are usually statements that if we use the parametrization by capacity in both models,

then the parametrized random curves in the two models have the same distribution. It is natural to expect that if we use the fractal variation to parametrize the curves in the two models, then these parametrized curves should have the same distribution. Assuming that the fractal variation of the scaling limit of the discrete model is just a constant times the natural parametrization, if we use the natural parametrization for the discrete model (times a suitable constant) and the fractal parametrization for the SLE, then the parametrized curves should have the same distribution. We will test this conjecture by considering random variables which depend on the parametrization of the curve.

The most obvious random variable to study would be to consider the point on the random curve at a fixed value of the parametrization (the natural parametrization for the discrete model, the fractal variation parametrization for SLE). We conjecture these two parametrizations are equivalent, but there is a non-trivial constant of proportionality between them that we would need to estimate. Instead, we will consider random variables that avoid the need to compute this proportionality constant.

In the following we use a superscript $'$ to indicate quantities defined in lattice models, while quantities without such a superscript will indicate quantities in SLE. For example, γ will denote an SLE curve while γ' denotes a random curve from the scaling limit of a discrete model. T will denote a value of the parameter in SLE where the parametrization is defined using the fractal variation along the SLE curve, and T' will denote a value of the natural parametrization in a discrete model.

We consider three of the discrete models in the upper half plane: the loop-erased random walk, the self-avoiding walk and the percolation exploration process. For these models we study the following random variables. Fix $R > 0$. First we consider the SLE curve. Look at the portion of the curve from its start at the origin until it first hits the semicircle of radius R . Let T be the fractal variation of this part of the curve. (Of course, T is random.) Let $\gamma(t)$ denote the curve parametrized by the fractal variation. So $\gamma(T)$ is the first point on the curve with $|\gamma(T)| = R$. Consider the point $\gamma(T/2)$. It can be thought of as the point that is halfway along the curve from 0 to $\gamma(T)$.

For the discrete models, let T' be the time when it first hits the semicircle. (This is the number of steps suitably scaled.) Let γ' denote the curve for the scaling limit of the discrete model and consider the point $\gamma'(T'/2)$. Then we expect that $\gamma(T/2)$ and $\gamma'(T'/2)$ have the same distribution. Let (X, Y) and (X', Y') be these two random points. We test the conjecture by comparing the distributions of X and X' and comparing the distributions of Y and Y' . We also write the points in polar coordinates, $X + iY = R \exp(i\Theta)$, and compare the distributions of R with R' and of Θ with Θ' .

The cumulative distributions of X, X', Y and Y' for the LERW, SAW and percolation are shown in Figs. 7, 8, and 9. Each figure contains four curves. For the LERW and SAW, the cumulative distributions of X and X' agree so well that the two curves are virtually indistinguishable in the figure. Likewise, the Y and Y' curves are virtually indistinguishable. So in Figs. 7 and 8 it appears there are only two curves. For percolation the difference between the cumulative distributions for percolation and SLE₆ are more noticeable, especially for Y and Y' . We do not fully understand this discrepancy, but believe it is caused by the difficulty of simulating SLE when $\kappa > 4$.

Table 1 shows the maximum difference between the cumulative distributions for the discrete model and SLE. With two exceptions these differences are on the order of one percent or less. The two exceptions are the Y, Y' and R, R' comparisons for percolation. We do not show any plots of the cumulative distributions of R, R', Θ or Θ' .

For the Ising model we consider a slightly different random variable since we cannot simulate the Ising model in a half plane. For the Ising model itself we take a rectangle of

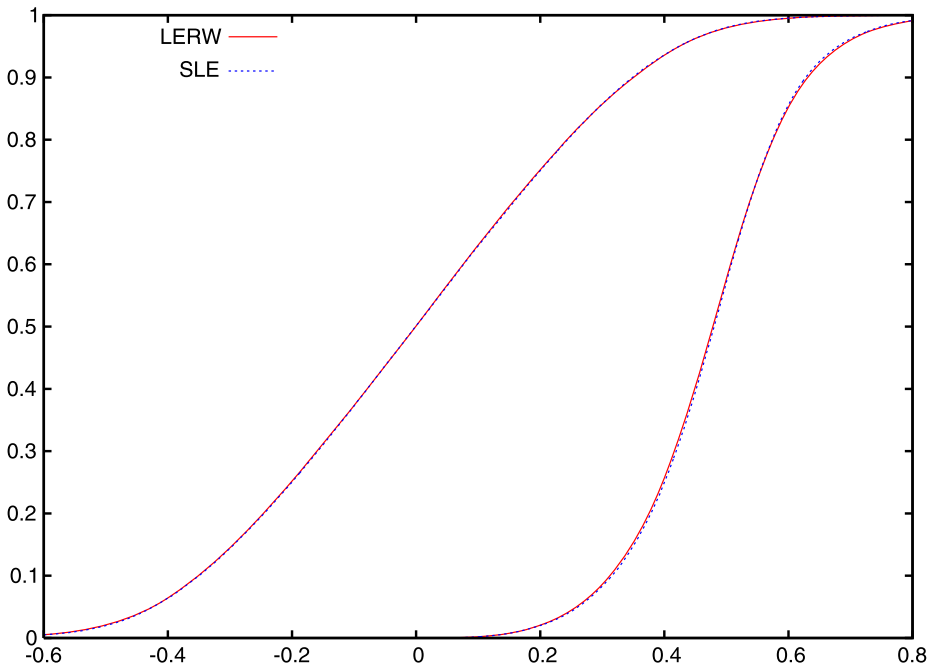


Fig. 7 (Color online) Cumulative distributions of X coordinate (left two curves) and Y coordinate (right two curves) of midpoint for LERW and SLE_2

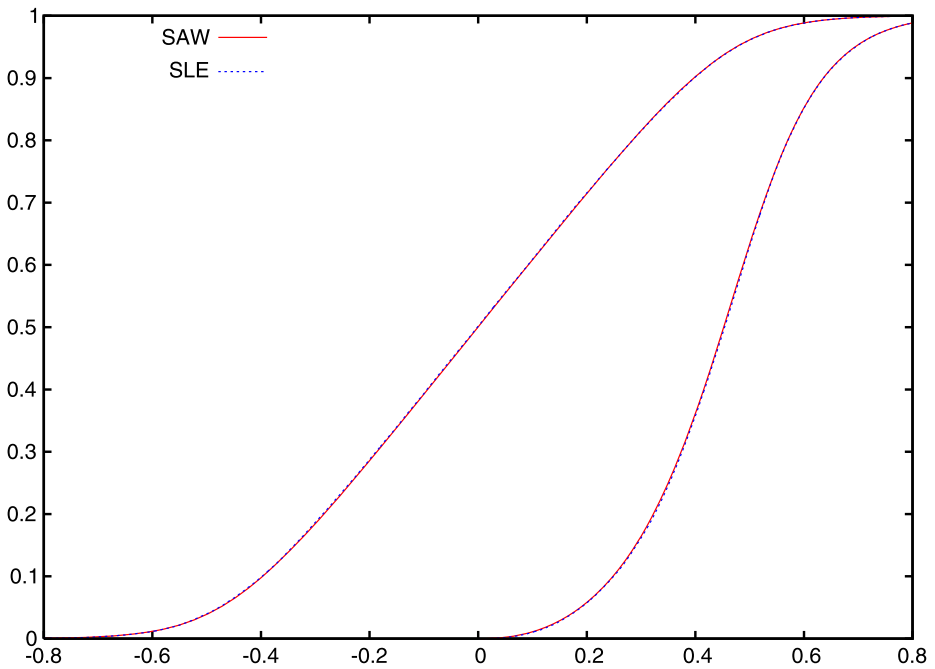


Fig. 8 (Color online) Cumulative distributions of X coordinate (left two curves) and Y coordinate (right two curves) of midpoint for SAW and $SLE_{8/3}$

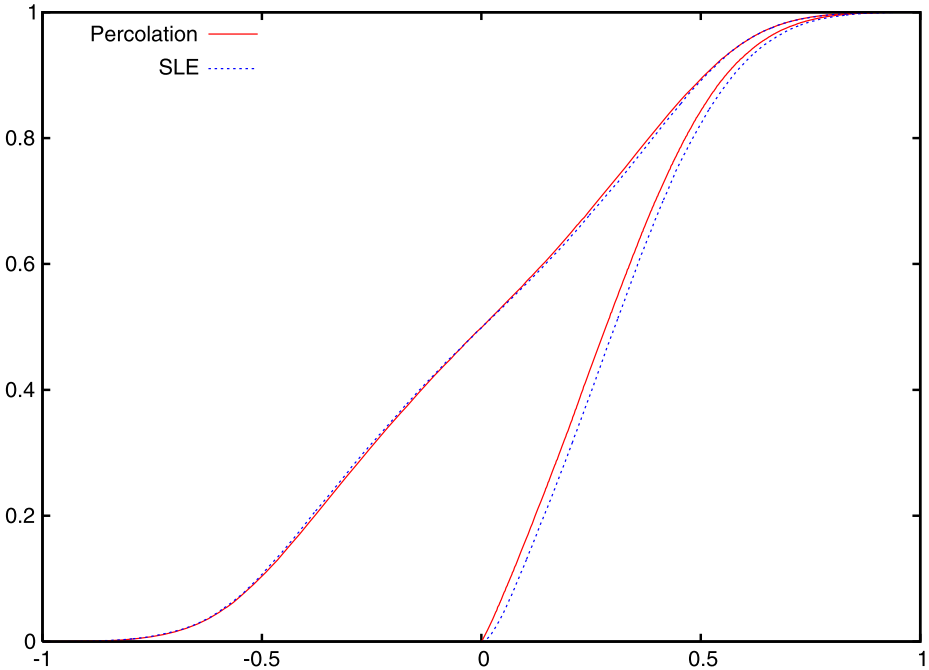


Fig. 9 (Color online) Cumulative distributions of X coordinate (left two curves) and Y coordinate (right two curves) of midpoint for percolation and SLE_6

Table 1 For a particular model and random variable, the table gives the maximum difference between the cumulative distribution function for the random variable in the lattice model and in SLE with the corresponding value of κ

	X	Y	R	Θ
LERW	0.003077	0.009211	0.007674	0.002425
SAW	0.002306	0.005238	0.007831	0.002808
Percolation	0.009293	0.040818	0.042395	0.013513
Ising	0.003686	0.005446	0.003371	0.003141

width W and height L with corners at $-W/2, W/2, -W/2 + iL$ and $W/2 + iL$ where W is several times as large as L . On the top and bottom sides the boundary conditions are -1 for boundary sites with negative x coordinate and $+1$ for boundary sites with non-negative x coordinate. We impose antiperiodic boundary conditions between the vertical sides. These boundary conditions force an interface from 0 to iL and approximate an infinite strip [1]. We let T' be the natural length of the Ising interface and consider $\gamma'(T'/2)$. We compare this Ising model with chordal SLE_3 in an infinite strip starting at 0 and ending at iL , where L is the width of the strip. We let T be the fractal variation of the entire SLE curve and consider the point $\gamma(T/2)$. Then as for the other models we compare the cumulative distributions of the x and y coordinates of the random points $\gamma(T/2)$ and $\gamma'(T'/2)$. Figure 10 compares the cumulative distributions of X, X', Y and Y' . Table 1 shows the maximum difference between their cumulative distributions, and those of the polar coordinates.

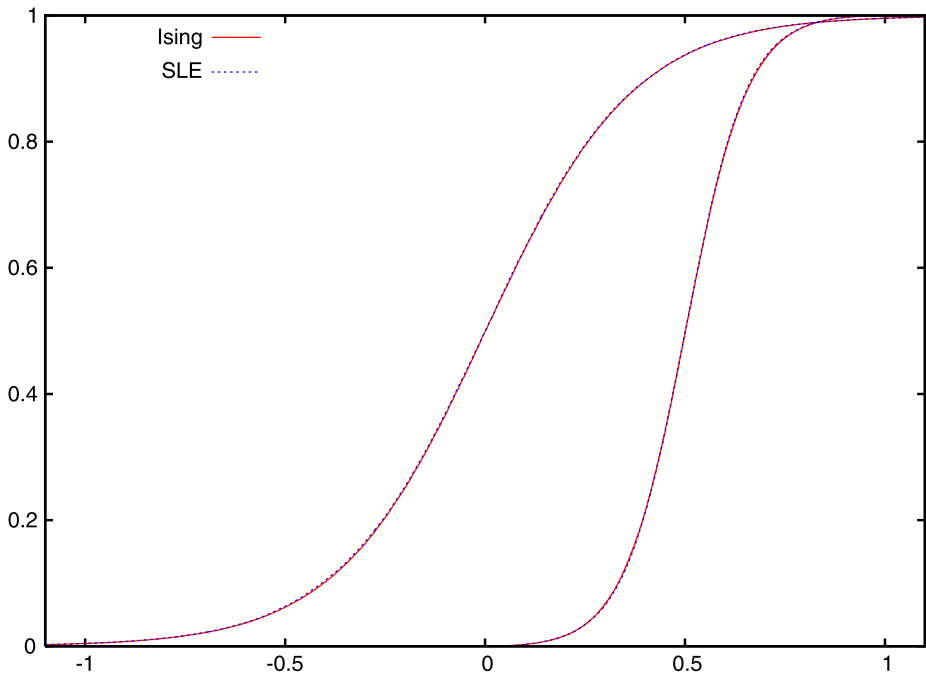


Fig. 10 (Color online.) Cumulative distributions of X coordinate (left two curves) and Y coordinate (right two curves) of midpoint for Ising and SLE_3

4 Simulation Details

In this section we provide some details about the simulations of both the lattice models and SLE.

To simulate SLE over the time interval $[0, t]$ we let $0 = t_0 < t_1 < t_2 < \dots < t_n = t$ be a partition of the time interval. The times t_k play a special role, but the random curves are still defined for all time. We replace the Brownian motion in the driving function by a stochastic process that equals the Brownian motion at the times t_k , and is defined in between these times so that the Loewner equation may be solved explicitly. There are several ways to do this. We interpolate the driving function in between the time t_{i-1} and t_i by a square root function. If g_t is the conformal map that takes the half plane minus the SLE hull back to the half plane, then this corresponds to approximating g_t by a sequence of conformal maps, each of which maps the half plane minus a linear slit back to the half plane.

The simplest choice of the t_i is to take a uniform partition of $[0, t]$. This does not work well since it produces points along the SLE curve that are far from being uniformly spaced. Instead we use an “adaptive” choice of the times [18]. Let $\Delta x > 0$ and let γ denote the SLE curve. If $|\gamma(t_i) - \gamma(t_{i-1})| > \Delta x$ we divide the time interval into two equal halves. We repeat this process until $|\gamma(t_i) - \gamma(t_{i-1})| \leq \Delta x$ for all the time intervals. Note that when we divide a time interval, we must use a Brownian bridge to choose the value of the random driving function at the time at the midpoint.

Computing points along the SLE curve requires evaluating the composition of approximately N conformal maps. If we compute N points along the curve, the total time needed for the computation will be $O(N^2)$. One can speed up the computation of the composition

by approximating the conformal maps by Laurent series. This leads to an algorithm that takes a time approximately $O(N^{1.4})$. This faster algorithm is explained in [10].

Our simulations of SLE for $\kappa = 2, 8/3$ and 6 are done in the half plane. We simulate each SLE curve until it hits the semicircle of radius 1. For $\kappa = 2$, we set $\Delta x = 0.002$ and generated 163,000 samples. For $\kappa = 8/3$ we set $\Delta x = 0.002$ and generated 132,000 samples. For $\kappa = 6$ we set $\Delta x = 0.005$ and generated 74,000 samples. The simulation for $\kappa = 3$ is somewhat different since we need the SLE in a strip, not the half plane. We generate this SLE in the half plane, and apply the conformal map that takes the half plane to an infinite strip, sending the origin to the origin and ∞ to i . We compute the distance between consecutive points on the curve for comparison with Δx after we have applied this conformal map. We stop the simulation when the tip of the SLE is within a distance Δx of i . We set $\Delta x = 0.005$ and generated 104,000 samples.

Lawler, Schramm and Werner proved that radial LERW converges to radial SLE_2 [15]. Radial refers to the fact that the random walk that one loop-erases begins at an interior point of the given domain and is conditioned to exit at a particular boundary point. One can also consider the chordal LERW in which the random walk begins at a boundary point and is conditioned to remain in the domain and exit at a particular boundary point. Zhan showed this model converges to chordal SLE_2 [24].

The LERW walk that we simulate is chordal LERW in the half plane from 0 to ∞ . This means that we take an ordinary random walk beginning at the origin and condition it to remain in the upper half plane. Then we erase the loops in chronological order. An ordinary random walk conditioned to remain in the upper half plane is known as the half plane excursion. It is trivial to simulate since it is just given by a random walk beginning at 0 with transition probabilities that only depend on the vertical component of the present location of the walk. If the site has vertical component k , then the walk moves up with probability $(k + 1)/4k$ and down with probability $(k - 1)/4k$. The walk moves to the right or left with probability $1/4$. (See, for example, Sect. 0.1 of [12].) The half-plane excursion is transient, i.e., each lattice site is visited by the excursion a finite number of times. This implies that the loop erasure makes sense. (For a recurrent walk all parts of the walk would eventually be part of a loop and so be erased.) Note, however, that if we take an infinite half plane excursion and only consider the first T steps and loop-erase this walk, the result will not agree with the loop-erasure of the full infinite excursion. A site which is visited by the excursion before time T may be erased by a loop formed after time T .

In practice there is no way to know if a visit to a site will be erased by some future loop without simulating the entire excursion. So in the simulation we do the following. We generate a half-plane excursion, erasing the loops as they are formed. We stop when the resulting walk has N steps. If α is small, then the distribution of our walk for the first αN steps will be close to the true distribution of the first αN steps of the LERW.

For the simulations comparing the distribution of the “midpoint” with the corresponding point in SLE_2 , we set $N = 10^5$ and generated 118,000 samples. We use a semicircle of radius $\rho N^{1/d_H}$ where $N = 10^5$ is the number of steps. We set $\rho = 0.1, 0.2, 0.4, 0.8$. If ρ is too large, the finite length effects will begin to appear. We find no evidence of these effects for $\rho = 0.1, 0.2, 0.4$, and good agreement between these cases. The finite length effects are seen when $\rho = 0.8$. The plots shown in Fig. 7 are for $\rho = 0.4$. We rescale all distances so that the radius of the semicircle is 1. For the simulations of the fractal variation we do not see any finite length effects and so we use the full 10^5 steps in the walk in computing the fractal variation. In these simulations we generated 132,000 samples.

The SAW in the upper half plane is defined as follows. Let N be a positive integer. We consider all nearest neighbor walks with N steps in the upper half plane which begin at

the origin and do not visit any site more than once. Then we put the uniform probability measure on this finite set of walks. We then let $N \rightarrow \infty$ to get a probability measure on infinite self-avoiding walks on the unit lattice in the upper half plane. Finally, we take the lattice spacing to zero. Lawler, Schramm and Werner conjectured that this scaling limit is $\text{SLE}_{8/3}$ [14]. They proved the existence of the $N \rightarrow \infty$ limit, but the existence of the limit as the lattice spacing goes to zero has not been established. Simulations of the SAW support their conjecture [8, 9].

The SAW in the half plane with a fixed number of steps may be simulated by the pivot algorithm, a Markov Chain Monte Carlo method [17]. We use the fast implementation of this algorithm introduced in [7]. For the SAW there is an issue similar to the LERW. The pivot algorithm produces the uniform distribution on the set of walks with N steps. But this is not the distribution of the infinite SAW in the half plane restricted to walks of length N . As with the LERW, we address this problem by simulating walks with N steps but then working with random variables that typically only depend on a relatively small initial part of the walk.

For the simulations comparing the distribution of the “midpoint” with the corresponding point in $\text{SLE}_{8/3}$, we set $N = 10^6$. We ran the pivot algorithms for 2×10^9 iterations of the Markov chain, and sampled the midpoint at each iteration. Of course, the resulting samples are far from independent. We use a semicircle of radius $\rho N^{1/d_H}$ with $\rho = 0.2$. For the simulations for the fractal variation we again used walks with $N = 10^6$ and ran the simulation for 2×10^9 iterations of the Markov chain, but we only computed the fractal variation every 10^3 iterations since this computation is relatively time consuming. Again, we computed the fractal variation of the full 10^6 step walk. Earlier Monte Carlo studies of the fractal variation in the SAW and the comparison of the SAW and $\text{SLE}_{8/3}$ using the fractal variation may be found in [11].

The Ising model we simulate is defined on a triangular lattice in a rectangle. Mixed boundary conditions are used on the top and bottom sides and antiperiodic boundary conditions on the vertical sides as described in the previous section. This forces an interface into the system that begins at the origin and ends at iL . Smirnov has announced a proof that the scaling limit of this interface is SLE_3 [21, 22].

We simulated the Ising model at its critical point with the Wolff algorithm [23]. The simulation was done in a rectangle of size $W = 2000\sqrt{3}$ by $L = 1500$. We ran the Wolff algorithm for 1.4×10^7 iterations, sampling the random variable every 100 iterations. The number of steps in the interface is random. We computed the fractal variation of the first ρL^{d_H} steps with $\rho = 0.3$ and 0.6 . The data plotted is for $\rho = 0.6$.

The percolation model we study is site percolation on the triangular lattice in the upper half plane, but we will describe it using the hexagonal lattice in the upper half plane. Each hexagon is colored white or black with probability $1/2$. The hexagons along the negative real axis are white and those along the positive real axis are black. This forces an interface which starts with the bond through the origin between the adjacent differently colored hexagons on the real axis. This interface is the unique curve on the hexagon lattice which begins at this bond and has all white hexagons along one side of the interface and all black ones along the other side. Smirnov proved conformal invariance for this model, and so the interface is SLE_6 [20]. See also Camia and Newman [4].

The percolation interface is the easiest of the lattice models to simulate. The key observation is that one should not generate the entire percolation configuration in a large rectangle but rather generate the configuration only as needed to determine the next step in the interface. Interfaces with several million steps can be generated in a few seconds. Note that unlike the LERW or SAW there is no finite length effect. If we generate interfaces with n

steps, they will have exactly the same distribution as the first n steps of interfaces of length N where $N > n$. The simulation was done with $N = 4 \times 10^6$ and 98,000 samples were generated. We used a semicircle of radius $\rho \times N^{1/d_H}$ with $\rho = 0.2$. For the simulations of the fractal variation, we also used $N = 4 \times 10^6$, but generated 206,000 samples.

5 Conclusions

Our main conclusion is that for the various lattice models (LERW, SAW, Ising and percolation), the fractal variation exists and is proportional to the natural parametrization of the random curves defined using the length of the lattice curves. Our Monte Carlo simulations support this conclusion in two ways. The first set of simulations show that the random variables $fvar(\gamma[0, t], \Delta t)$ converge to a constant as Δt goes to zero. The second set of simulations support the conclusion in a more indirect way. If the conclusion is true for a lattice model and the scaling limit of that model is SLE $_{\kappa}$, then the fractal variation of the SLE curves should exist and provide a way to parametrize the SLE curve that corresponds to the natural parametrization in the discrete model, up to a constant. Our second set of simulations tested this by comparing the distribution of random variables that depend on the parametrization of the curve.

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Note Added in Proof A recent preprint by Greg Lawler considers several possible definitions of the natural parametrization of SLE [13].

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